Experimental Establishment of the Relativity of Time

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None of the fundamental experiments on which the restricted principle of relativity is based requires for their explanation that the classical concept of absolute time be modified; the present experiment was devised to test directly whether time satisfies the requirements of relativity. It depends on the fact that if a pencil of homogeneous light is split into two components which are made to interfere after traversing paths of different length, their relative phases will depend on the translational velocity of the optical system unless the Lorentz-Einstein transformation equations are valid. Hence, such a system at a point on the earth should give rise to an interference pattern which varies periodically as the velocity of the point changes in consequence of the rotation and revolution of the earth. The effect to be expected for a small velocity is so very small that it has been necessary to devise a special source of light, an interferometer of great stability and a refinement of the technic of measuring displacements in the interference pattern. With the apparatus finally employed, we have shown that there is no effect corresponding to absolute time unless the velocity of the solar system in space is no more than about half that of the earth in its orbit. Using this null result and that of the Michelson-Morley experiment we derive the Lorentz-Einstein transformations, which are tantamount to the relativity principle.

AMONG the several classical experiments which suggested the restricted principle of relativity there appears to be none in which any question as to the nature of time is involved. That is, in any of them, time as indicated by an ideal clock moving with the earth might be related in any way to that indicated by a hypothetical fixed clock without at all affecting their results, at least insofar as can be inferred from such theories of the experiments as we are at present able to construct. In experiments such as those of Rayleigh and Brace, of Trouton and Noble, and of Fizeau, all of which yielded null results, there is present the theoretical difficulty that unknown properties of matter are involved. The Michelson-Gale experiment gives a positive result, which is consistent with the concepts of either relative time or absolute time. In fact, it seems that the only experiment heretofore reported that permits of any definite interpretation is that of Michelson and Morley; and the null result of this experiment is completely explained if we suppose that space dimensions in the direction of motion are contracted by an amount depending upon a suitable function of velocity; so here, too, no question as to time is raised. Hence, although such experiments have suggested the relativity theory, they do not form a sufficient basis for the logical derivation of it.

It appears, then, that the theory has needed confirmation, particularly in its most revolutionary aspect; i.e., its denial of a significance for absolute time. Such confirmation has been obtained in the work reported in this paper, and by combining our results with those of the Michelson-Morley experiment, we derive the Lorentz-Einstein transformations which are well known to embrace the whole theory.

The principle on which this experiment is based is the simple proposition that if a beam of homogeneous light is split at a half-reflecting surface into two beams which after traversing paths of different lengths are brought together again, then the relative phases of the superposed beams will depend upon the velocity of the apparatus unless the frequency of the light depends upon the velocity in the way required by relativity. Furthermore, the phase-difference can be made to determine the positions of fringes in an interference pattern, so that by measuring these positions for various velocities of the system, the question whether the frequency follows the relativity requirement can be decided. The variation of the velocity of the system comes about because of the motions of rotation and revolution of the earth.

The theory of this experiment requires the following two assumptions: (a) There exists at least one coordinate system in which Huyghen's principle is valid and the velocity of light is the same in all directions. This assumption is unobjectionable from the standpoint either of relativity or of any plausible hypothesis involving an ether; for relativity, it is true for all uniformly moving systems, and in the latter case for any system at rest in the ether. (b) The Michelson-Morley experiment indicates that a system moving with uniform velocity v with respect to such a system has dimensions in the direction of motion contracted in the ratio $[1-v^2/c^2]^{1/2}$ as compared to dimensions in the fixed system, while dimensions perpendicular to this direction are unchanged. This is in part assumption, for although there can be little doubt that the experiment yields a strictly null result, nevertheless it actually shows only that dimensions in the direction of and perpendicular to the motion are in the ratio mentioned; either of these dimensions might be any function of the velocity so long as that ratio is preserved.

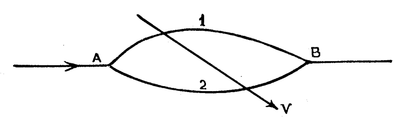

Fig. 1.

Let us consider one such system S', and suppose that a system S (attached, for instance, to the surface of the earth) moves practically uniformly with velocity v with respect to it. In S is set up an arrangement for producing interference; i.e., one in which a pencil of homogeneous light is divided as mentioned above into two pencils which are recombined after traversing paths of different lengths. We can simplify the discussion by treating the general case instead of the particular arrangement used in the experiment, and by adopting a rule regarding expressions for the distances, angles and times in system S that will be of interest; i.e., the magnitudes of these quantities will be expressed by unprimed letters when they are referred to standards moving with S, and by the same letters primed when referred to standards fixed in S'.

The path of a typical ray with respect to S can be represented schematically as in Fig. 1 where the ray coming from the left is divided at A into rays 1 and 2 which recombine at B. The courses of the same rays with respect to

S' are evidently determined by the requirement that to each element ds' of a ray is to be added an elementary vector vdt' where dt' is time required for light to traverse the element, and c is the velocity of light with respect to S'. The length of the resulting element is evidently cdt'; hence $c^2(dt')^2=(ds')^2+v^2(dt')^2+2v\,ds'dt'\cos\theta'$. Hence,

$$dt' = \frac{ds'}{c(1-\beta^2)}[\beta\cos\theta' + (1-\beta^2\sin^2\theta')^{1/2}] \tag{1}$$

where $\beta=v/c$ and θ' the angle between v and the element ds'.

If for the moment we consider a set of rectangular coordinates in S and S' with corresponding axes parallel and x-axes parallel to velocity v, we have from assumption (b)

$$ds' = [(dx')^2+(dy')^2+(dz')^2]^{1/2} = [(dx)^2(1-\beta^2)+(dy)^2+(dz)^2]^{1/2}$$

$$= ds\left[1-\beta^2\left(\frac{dx}{ds}\right)^2\right]^{1/2} = ds(1-\beta^2\cos^2\theta)^{1/2}$$

$$\cos\theta' = \frac{dx'}{ds'} = \frac{dx(1-\beta^2)^{1/2}}{ds(1-\beta^2\cos^2\theta)^{1/2}} = \cos\theta\left(\frac{1-\beta^2}{1-\beta^2\cos^2\theta}\right)^{1/2}$$

$$\sin^2\theta' = \sin^2\theta/(1-\beta^2\cos^2\theta).$$

When these expressions are substituted in Eq. (1), it reduces to

$$dt' = [ds/c(1-\beta^2)^{1/2}](1+\beta\cos\theta), \tag{2}$$

the right side of which equation involves only quantities referred to standards moving with S. The time for light to traverse the whole ray AB along path 1 is therefore

$$t_1' = \int_1 dt' = 1/c(1-\beta^2)^{1/2}\int_1 (1+\beta\cos\theta)ds$$

and a similar expression holds for path 2. Hence difference of time for the two paths is

$$t_1'-t_2' = 1/c(1-\beta^2)^{1/2}\left\{\int_1 ds - \int_2 ds + \beta\left[\int_1 \cos\theta\,ds - \int_2 \cos\theta\,ds\right]\right\}.$$

The term in brackets multiplied by β vanishes, since in order to interfere the rays must intersect, and therefore their projections on the line joining A and B are equal; these projections are the integrals in brackets. Hence

$$t_1'-t_2' = (s_1-s_2)/c(1-\beta^2)^{1/2} = \Delta s/c(1-\beta^2)^{1/2},\ \text{say},$$

and the number of waves corresponding to this difference of time is

$$n = \nu'(t_1'-t_2') = \nu'\Delta s/c(1-\beta^2)^{1/2} \tag{4}$$

where ν' is the frequency of the light employed as measured by an observer in S'. This number n is seen to be independent of orientations, lengths and dispositions of paths, but to depend upon difference of path-lengths, the relative velocity of S and S' (through β) and the frequency.

The foregoing treatment is strictly valid only if the moving system is regarded as not subjected to forces, but is undoubtedly sufficient for the purpose in the small constant field of gravitation and acceleration at the surface of the earth. Moreover, although the rotation of the apparatus with the earth involves a slight effect on the time difference computed above (whether regarded from the standpoint of relativity or classical theory), it turns out to be altogether negligible in amount. This effect is a function of rotational velocity, not of orientation of apparatus.

We have now to consider the effect of a change in the velocity v on the number n expressed by Eq. (4). In that equation c is evidently a constant, while the difference Δs, because it is referred to standards moving with the system, is constant unless the courses of the rays between the points of separation and recombination are dependent on the velocity; that this is not the case can be shown by Huyghens' principle. A direct consequence of this principle is that the course of the ray is determined by the condition that the time required for traversing the path is a minimum compared with the time for any neighboring path. Now, Eq. (3) expresses the time in terms of coordinates moving with S, and if minimized in the usual way would yield the equations of the paths. For the present purpose, however, it is unnecessary to carry out this operation. Rewriting (3) we have

$$t' = 1/c(1-\beta^2)^{1/2}\left\{\int_A^B ds + \beta\int_A^B \cos\theta\, ds\right\}.$$

It will be observed that although the expression involves the velocity of the moving system, nevertheless the course of the ray is quite independent of it. That this is so is evident from the following considerations: the second integral is equal to the projection of the path on the line joining A and B, and being the same therefore for all paths, cannot contribute to the determination of the minimizing path. The first integral is expressed in terms of distances referred to standards moving with the system and so is independent of the velocity. Hence the actual courses of the rays, which are got by minimizing integrals of this form, are independent of the velocity, and Δs is a constant. This proof is essentially that of Lorentz extended by the inclusion of the contraction hypothesis.

The quantity ν' in Eq. (4) is the only one whose possible variability with velocity remains to be considered. From the standpoint of relativity, $\nu' = \nu(1-\beta^2)^{1/2}$ where ν is the constant value of the frequency which would be determined with standards moving with S; this value of ν' would evidently make n a constant. Furthermore, it will be shown later that insofar as the atom is to be regarded as a typical clock, the Lorentz-Einstein transformations can be derived from this relationship and assumption (b). If, on the other hand, $\nu' \neq \nu(1-\beta^2)^{1/2}$ these transformations do not apply and it turns out that there exists but one system S' satisfying assumption (a); this unique system would be the absolute reference frame postulated in the classical ether theory. In this case n is evidently a function of the velocity of S with respect to the absolute reference frame. Evidently, then, the relativity hypothesis can be tested by determining whether n is constant as v changes in consequence of the motions of rotation and revolution of the earth.

For the present purpose the total velocity of the apparatus can be got by adding vectorially a presumably constant velocity v_0 of the sun, the orbital velocity v_1 of the earth and the circumferential velocity v_2 due to the rotation of the earth (taking account of latitude). Its square can be reduced to

$$v^2 = v_0^2 + v_1^2 + v_2^2 + 2v_\alpha v_1 \sin(\theta_1 - \omega_1) + 2v_\beta v_2 \sin(\theta_2 - \omega_2) + 2v_1v_2 \cos(\theta_1 - \theta_2),$$

where v_α is the projection of v_0 on the orbital plane, v_β is projection of v_0 on the equatorial plane, ω_1 and ω_2 are constants related to the direction of v_0, and θ_1 and θ_2 are angles expressing the position of the earth in its orbit and its orientation on its axis with respect to the fixed-stars.[1] This procedure assumes only that the fixed-star system has no great angular velocity with respect to the fundamental system S'; there is an unimportant approximation in the last term.

In order to get an idea of the magnitude of the effect that might be expected, let us assume that $\nu' = \nu$ and replace ν by c/λ; then (4) becomes $n = \Delta s/\lambda(1-\beta^2)^{1/2}$. Expanding this, ignoring terms in β above second degree, substituting for the velocity from the expression above, and gathering constant terms into one,

$$\begin{aligned} n &= (\Delta s/\lambda)(1 + \tfrac{1}{2}(v^2/c^2) + \cdots) \\ &= (\Delta s/\lambda c^2)\left[v_\alpha v_1 \sin(\theta_1 - \omega_1) + v_\beta v_2 \sin(\theta_2 - \omega_2)\right] + \text{a constant} \\ &= \delta n + n_0. \end{aligned} \tag{5}$$

Here the variable part of n is represented by δn and the constant by n_0 and we assume v_α and v_β to be large compared with the orbital and circumferential velocities v_1 and v_2. Hence δn should be proportional to the sum of a term with a period of a year and one with a period of a sidereal day.

In performing the experiment, we wish, of course, to make δn as large as possible. The only factor that can be controlled is the ratio $\Delta s/\lambda$, the largest feasible magnitude of which is a measure of the homogeneity of the light. For various reasons the most suitable light seems to be the mercury line of wave-length 5461. With this, sufficiently clear interference fringes could be got when Δs was as large as 318 mm (the value finally used) and on substituting this in the expression for n it turns out that the rotation of the earth would produce a daily variation of a thousandths of a fringe for 200 km per

[1] More specifically, θ_1 is the angle between the projection of v_0 on the orbital plane and a direction in that plane determined by the angle ω_1 which depends on the position of the earth in its orbit or the time of year at which θ_1 is taken as zero. Similarly, θ_2 is the angle between the projection of v_0 on the equatorial plane and a direction in that plane determined by the angle ω_2 which depends on the time of day at which θ_2 is taken as zero. In the reduction of the data, the θ's are taken as zero at the beginning of each run, so the ω's depend on the times of starting runs. In the comparison and final summary of data, the θ's are of course referred to the same sidereal time.

second, while the orbital motion would produce the same variation in six months for 3 km per second.

Because of the probable minuteness of these effects it was necessary to contrive new ways of detecting them. In the rather complicated method first proposed[2] the phase variation would show itself in the rotation of the plane of polarization of a beam resulting from the superposition of two oppositely circularly polarized beams. This scheme, although theoretically capable of great precision, was abandoned in favor of the much simpler one finally employed. In the latter, ordinary interference rings were formed and photographed, and the problem became one of measuring very small changes in the diameters of the rings. It was satisfactorily solved by devising a special comparator which will be discussed later.

Evidently, it was necessary to take every precaution to keep the experimental conditions constant; we were able, in fact, to reduce the average daily periodic error in $\Delta s/\lambda$ to about two parts in 10^{10}. This great stability was attained mainly by using interference apparatus made almost entirely of fused quartz and kept in a vacuum at a temperature constant to within about a thousandth of a degree. The apparatus was furthermore (partly accidentally) compensated for temperature to such an extent that one degree change produced a shift of only about a hundredth of a fringe. The vacuum was employed as simplest way to eliminate variations in pressure, which would have caused variations in index of refraction of optical paths, and, by mechanical action, variations in lengths of paths.

Several disturbing factors producing spurious effects had to be dealt with. Perhaps the most troublesome was the variability in density of the photographs due (in the earlier green-sensitive plates) to rapid aging which affected the emulsions in varying degrees. Since the photographic effect of light is not proportional to its intensity, it follows that a spurious displacement of an interference pattern of the type used is to be expected if the density of the photographs is not constant. The methods adopted to eliminate this and other difficulties are discussed elsewhere in the paper.

Apparatus and Experimental Procedure

The general arrangement of the experimental apparatus is sketched in Fig. 2. Light from source S passes through a small circular opening in screen S_1, is rendered approximately plane-parallel by lens L_1 is dispersed in direct vision prism P and the green (λ5461) image of first opening is focused over a second one in screen S_2 by lens L_2. The water-cell C is to absorb stray heat radiation. The green light from second opening is polarized by nicol prism N so that the electric vector is horizontal, and then enters the vacuum chamber V through a window and is concentrated by lens L_3 to the extent required to produce the greatest intensity in interference pattern. The light is then split into two pencils at the half-reflecting mirror M_1 which is inclined at such an angle (Brewster's angle) that reflection of the polarized light occurs only at its platinized face; the faces of the compensating plate M_4 are equally

[2] Kennedy, Phys. Rev. **20**, 26 (1922).

inclined. Hence no stray (non-interfering) light can be superposed at these faces on the two pencils from M_1; these pencils are reflected by mirrors M_2 and M_3 back to M_1, at which one is partially transmitted and the other partially reflected through lenses L_4 and L^5 which focus the light as a system of interference rings on a wide horizontal slit just in front of a photographic plate in the holder H. The slit is 5 or 6 mm wide, so the plate receives a symmetrical central section of the interference pattern of that width. The plate is held by a spring in the holder lightly against the metal tube T which is sealed against the window W of the vacuum chamber. Most of the length of the tube as well as the vacuum chamber is within the tank V containing

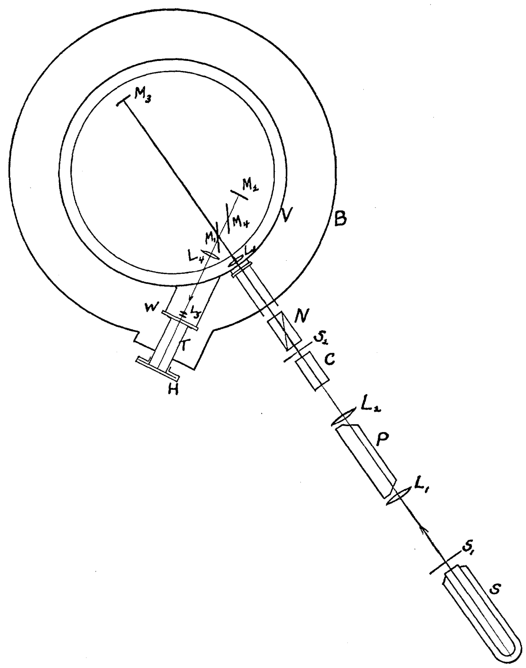

Fig. 2.

water at a temperature constant to within less than 0.001°C; hence slight variations in room temperature cannot affect focusing and thereby diameters of rings. The plate holder, which is kept from contact with the vacuum chamber in order to preclude the possibility of jarring the latter when the holder is operated, is arranged to let the plate slip down two slit-widths automatically every half hour. On each plate six photographs are taken consecutively in this way, and twelve hours after the start of the first series six more are taken in the spaces left vacant during first exposure of the plate; hence the developed plate will contain a series of photographs alternately taken twelve hours apart. The purposes served by this procedure will be explained later. Four such plates are taken during a day's run.

The temperature of water-bath was easily kept nearly constant for many weeks in succession. The temperature was chosen only slightly above that of room, the water was circulated continuously and the mercury-toluene thermostat was arranged to control the potential of the grid of a vacuum tube which actuated the relay in the heating circuit—in this way only a minute current is broken at the mercury surface and it does not become contaminated with a film of oxide. The optical part of the apparatus was enclosed in a small dark room within a larger one. The temperature of the inner room was kept constant to within a few hundredths of a degree, that of outer room to within about a tenth.

The interference apparatus consisted essentially of a set of four interferometer plates of the best quality obtainable, mounted on a circular fused quartz base 28.5 cm in diameter by 3.8 cm thick. The method of mounting the plates is perhaps worth describing: the support of each plate was cut from a flat plate of fused quartz, and fused to a tapered plug of the same material which, after being ground to fit a tapered hole in the base, was etched away

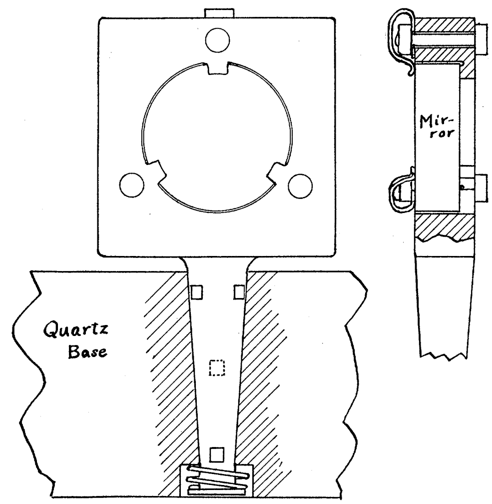

Fig. 3.

over the whole conical surface except in four spots of two or three square millimeters area, which were therefore the sole points of contact of the plug with the base. This procedure was necessary to insure a definite fixed position of plug, since if it were merely ground into the plate it would probably fit the hole only in a region near its middle. The positions of the bearing spots are indicated by the small squares in Fig. 3, the dotted one being on the opposite side of plug from the others. The plugs were held down by light springs as shown in the figure. The end mirrors were circular etalon plates 25 mm in diameter. Their supporting frames were fashioned so as to provide three projections of quartz against which the mirror was held by a light spring opposite each projection. The faces of the projections were ground flat and so as to be very nearly in a vertical plane when the frame is in place on the base. Final adjustment of mirrors was made by rotating them about their horizontal axes; it will be evident that in this way (because of slight inclination to each other of the faces of the mirror) a very fine adjustment can be made. It was sufficient simply to rotate the mirrors with the unaided fingers, to correct for the departure from the vertical, while viewing the interference rings with a telescope. The reflecting surfaces were of platinum applied by cathode deposition. It was impossible to use silver for the purpose because traces of mercury vapor in the vacuum chamber would quickly dissolve it. The light lens system which formed the rings on the photographic plate was attached to the base by means of invar plugs similar to those described above.

The quartz base rested on a piece of uniform velour, the back side of which was cemented to a heavy flat brass plate which was supported in an accurately horizontal position at three points. Each fiber of the nap of the velour thus served as a tiny spring so that the weight of the quartz plate was evenly distributed; this is important, since a fused material of this sort is essentially only semi-solid. The friction between the velour and the rough bottom face of the base sufficed to hold the latter accurately in position.

In order to produce interference under the existing condition of large difference of paths of the two beams, the image in the half-reflecting mirror of the face of either end-mirror must be nearly parallel to the face of the other end-mirror; it will be shown that such an adjustment of the mirrors gives rise to a pattern consisting of a series of concentric circular rings. In order that the effective diameter of each ring may be sensibly independent of accidental variations in distribution of light intensity over the faces of mirrors and with respect to direction in the beam, it is necessary to make this parallelism very accurate. The accuracy of adjustment could be tested by the simple procedure of moving a broad slit in various directions across the pencil incident on the half-reflector while the rings were observed in a telescope or photographed; when the diameters of rings were constant for all positions of slit the adjustment was the best obtainable.

The particular spectral line employed in the experiment was chosen on basis of several requirements. As has been pointed out, it must be capable of producing interference with large path-difference; it must also be entirely controllable as to intensity, the intensity must be fairly large, and the line must be easily separable from adjacent ones. On the whole, these conditions seemed best satisfied by the line $\lambda 5461$ of mercury. The homogeneity of any light is roughly proportional to the inverse square root of absolute temperature of source; hence the first source employed was a water-cooled mercury arc. This produced excellent interference rings, but it was soon noticed that their diameters depended on the part of the arc from which the light was taken; this suggests a Doppler effect due to motions of evaporating molecules from the hot liquid surface where the arc was brightest. Such an effect due to velocities variable by only a few centimeters per second would evidently be objectionable in view of the stability required.

The source finally used was an electrodeless discharge in unsaturated mercury vapor. The tube is sketched in Fig. 4. The inner tube in which the discharge took place was connected to a continuously operating pumping system through a capillary tube (heated to prevent condensation in it) of such length and diameter as to keep the pressure of the vapor just below that of saturated vapor at the existing temperature. The vapor was supplied by the mercury well at the rear of tube, and the small amount escaping through the capillary would condense and return by way of the other vertical tube. The temperature of the source, and thereby the pressure of vapor, were kept constant by means of carbon-tetrachloride in the jacket surrounding the inner tube; the liquid was maintained at its boiling-point by heat from the discharge, and its vapor was condensed and returned by the water-cooled condenser connected to top of jacket. It will be evident that with the discharge occurring at some distance from the mercury well, first-order Doppler effects would be eliminated since no mercury condenses in the forward part

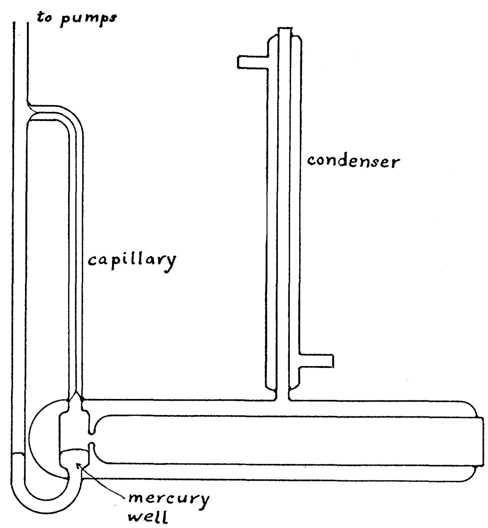

Fig. 4.

of the tube and therefore velocities of vapor molecules are on average same in all directions. Electrical energy was supplied by a coil of some thirty turns of wire around the outside of the jacket, in which oscillations of 20 meters wave-length were produced by a 75-watt transmitting tube. This discharge produced a uniform steady glow over nearly the whole diameter of the inner tube, and the interference rings were completely free from the fluctuations in brightness and diameter which were visible with the ordinary arc. During a run, and for some time in advance of it, the tube was kept in continuous operation in order that all conditions should be steady. It was found that the frequency of the light depended on the temperature of the cooling liquid and the voltage applied to oscillator, so these factors had to be closely controlled. These effects probably arise from the complicated structure of the green line; its "frequency," as inferred from the interference pattern, is of course a sort of mean of the frequencies of its components, weighted according to their intensities. It is to be mentioned that each of several attempts to use sealed-off tubes failed; after a few minutes of operation with such tubes the rings would disappear, presumably because the oscillatory discharge readily excited a green band in traces of oxygen which probably remain in tube.

In view of the theorem of Lorentz previously discussed, the usual theory of interference for stationary systems can be applied directly to the present situation. In Fig. 5, A represents the surface of one end-mirror and B the

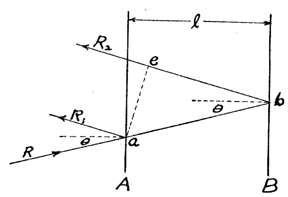

Fig. 5.

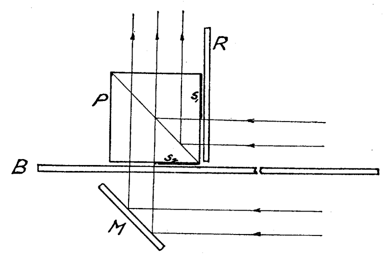

Fig. 6.

image of the other at distance l from A. Since A and B are parallel, the ray R impinging on both at angle θ produces on reflection the two parallel rays R_1 and R_2. If these are brought to a focus, the difference between the lengths of their paths will evidently be $ab+be$. Now

$$ab = l/\cos\theta,\ be = ab\cos 2\theta.\ ab + be = (l/\cos\theta)(1+\cos 2\theta) = 2l\cos\theta.$$

For constructive interference, this path-difference must contain an integral number of waves; hence the cones of rays for which $2l\cos\theta_i(i=1, 2, 3, \cdots)$ equals a series of consecutive integers[3] can be brought to a focus as a series of concentric rings of radii $r_i=(\sin\theta_i)/k_1$, where k_1 is a constant depending on magnification of lens system producing the interference pattern. Now $ab+be$ is the quantity Δs in Eq. (4); hence $\Delta s_i=2l\cos\theta_i=2l(1-k_1^2r_i^2)^{1/2}$ and $n=2\nu'l(1-k_1^2r_i^2)^{1/2}/c(1-\beta^2)^{1/2}$. It is convenient to consider only the central ray, and to express its phase in terms of the radii of the rings. For this ray $r=0$, so

$$n = 2\nu'l/c(1-\beta^2)^{1/2} = n_0+\rho \tag{6}$$

where n_0 is an integer and ρ a fraction. In general, for constructive interference $n=n_0-i$. Then

$$n_0 - i = [2\nu'l/c(1-\beta^2)^{1/2}](1-k_1^2r_i^2)^{1/2} = (n_0+\rho)(1-k_1^2r_i^2)^{1/2}.$$

$$\therefore\quad \rho = (n_0-i)/(1-k_1^2r_i^2)^{1/2} - n_0 = n_0 - i + \tfrac{1}{2}(n_0-i)k_1^2r_i^2 - n_0\cdots$$

$$= (k/2)r_i^2 - i, \tag{7}$$

approximately; here k is a new constant. The approximations are justified since n is of order $10^5\times i$ and k_1r_i has a maximum value of about 10^{-2} for the rings measured. From Eq. (7) we find on differentiating

[3] When the distances are expressed in wave-lengths.

$$\delta\rho = kr_i\delta r_i = kr_j\delta r_j = \cdots . \tag{8}$$

If measurements $\overline{\delta r_i}$ of the values of the variations in r_i are made for each of a number of rings of orders m to p, the mean value of $\delta\rho$ computed from them is

$$\delta\rho = \frac{k}{p - m + 1} \sum_m^p r_i\,\overline{\delta r_i}. \tag{9}$$

It will be convenient to have $\delta\rho$ in another form. In the final summary of data there are many values of $\delta\rho$ to be averaged for each value of the hypothetical velocity. It is clear, then, that the final average will be unaffected if we replace the variations δr_i in (8) by their individual measured values $\overline{\delta r_i}$, so that $r_i\overline{\delta r_i} = r_j\overline{\delta r_j}$. Multiplying and dividing the right side of Eq. (9) by $\sum_m^p 1/r_i$ it becomes

$$\delta\rho = \frac{k}{\sum(1/r_i)} \sum\overline{\delta r_i} \tag{10}$$

when the expressions $(r_i/r_j)\overline{\delta r_i}$ that appear in the product are replaced by $\overline{\delta r_j}$. Since we are dealing with extremely small variations in the radii, the radii can be measured and $\Sigma 1/r_i$ computed once for all for a given adjustment of apparatus; then the variations $\delta\rho$ are simply proportional to the sums of the variations in the several radii. This possibility greatly expedites the labor of measurement of plates; the way in which it was employed is discussed in connection with the description of the comparator designed for the purpose. It should be remarked that in this procedure insufficient weight is given to the somewhat greater precision of measurements on the larger, sharper, rings; however the final weighting of data is based on mean deviations of the computed values of $\delta\rho$, and the conclusions as to precision are not vitiated by this approximation.

The principle of the comparator is as follows:

A diametral section of each photograph to be measured is made to appear juxtaposed with a similar section of a nearly identical photograph which is used as a standard of reference for the whole series. In this way very small differences between reference and measured plates reveal themselves. The juxtaposition is along a diameter of each of the systems of concentric rings, and the comparison is made by moving the standard until one side of a ring on one plate appears to be continuous with the corresponding ring on the other plate, and then noting the distance along the line of demarcation which the standard must be moved in order to make the other sides of same rings coalesce similarly. This distance is evidently the difference between the diameters of the two rings. On the shaft of a fine micrometer screw which moves the reference plate, and concentric with it, is mounted a graduated slip-ring arranged so as to be held stationary when the portions of interference rings on one side of center are matched, and to rotate with the screw when the matching is done on the other side of center; the angle through which slip-ring is rotated during settings on a number of rings is thus evidently proportional to the sum of the differences of their diameters from those on corresponding rings on reference plate. Hence in view of Eq. (10) a single reading (of this angle) summarizes the measurement of the whole exposure. Nine or ten rings, alternately dark and light, and near the center, were usually measured.

The device is diagrammed in Fig. 6; there P is a pair of similar right-angled prisms cemented together on their diagonal faces (one of which is half-silvered) and mounted on a carriage which can be moved by the micrometer screw on ways perpendicular to the section represented. The heavy lines s_1 and s_2 represent thin metal strips covering half the right and bottom faces of the prism combination; the lower edge of s_1 and left edge of s_2 are ground accurately straight and the strips are cemented to the prisms in such a way that the image of the former edge in the diagonal mirror exactly coincides with the latter. After traversing a water-cell a beam of light from right of figure passes through the reference plate R, which is mounted on the carriage and has its emulsion side in contact with screen s_1 along a diameter of the ring system; another beam, by way of mirror M, illuminates a similar part of the photograph on the plate B which is to be measured, and both parts are viewed from above through a lens system magnifying about four times. The latter plate is held by springs against stops which fix position of the emulsion side regardless of thickness of plate and of course at such a distance as to eliminate parallax. The exposures can be compared in turn by sliding the plate to right or left of diagram (toward or away from operator). Since the ring system may not be exactly circular and also in order to expedite placing the plates in position for comparison, a sharp notch was cut in each end of the slit behind which the plate is held during exposure; this leaves a sharp point at each end of the photograph which serves for setting accurately along the same diameter. It is to be noted that the comparator is automatically compensated for temperature (both reference and measured photographs being on same material); this compensation was not particularly important for the present purpose because the scheme of interleaving photographs taken twelve hours apart secured the same result.

So accurately and quickly can the settings be made that the measurement of a photograph can be made after some practice with a probable error of a thousandth of a fringe (i.e., a thousandth of the shift that would be produced by changing path-difference by one wave-length) in about five minutes. The labor of comparing the 48 exposures comprising a day's run is thus not great. It was particularly desirable to be able to make rapid measurements during the numerous preliminary adjustments of apparatus, tests of effects of varying the several experimental conditions, etc.

Two precautions were taken in order to keep the operator from being influenced in making settings on the comparator. The slip-ring, on which could be read the average differences of the diameters at any stage of comparison of a particular exposure, was kept covered until the final setting was made, thus preventing unconscious corrections during the later settings. Also, the plates were marked in such a way that the operator was in complete ignorance of times of day at which they were exposed; not until a full day's readings were finished were they arranged in chronological order for computation.

Data and Results for Daily Effect

It was intended when the experiment was proposed to look chiefly for an effect of a change of velocity due to the orbital rather than the rotational motion of the earth. However with the first apparatus constructed, in which the mirrors were mounted in invar frames, it was found impossible to eliminate a slow, rather irregular variation in the interference pattern which would have masked the effect sought; hence it was decided to concentrate on the possible rotational effect. Three series of data were taken with this apparatus (in April and October, 1929 and January, 1930); after an interruption of over a year, during which the apparatus was rebuilt in its final form, three more series were taken in May, July and August 1931. The same form of light source was used in all six series. A large amount of data previously obtained with the water-cooled arc and under less carefully controlled conditions are ignored in the summary because of necessity of applying doubtful corrections to it. No corrections have been applied to the data here presented. Where results of the several series are combined, they are weighted in accordance with the usual theory of errors in terms of probable errors computed from the mean deviations.

Each of the series extended over a period of only a few days; during such a time we may regard sin $(\theta_1-\omega_1)$ in Eq. (5) as virtually constant. From this equation and Eq. (6), $\delta n=\delta\rho+$a constant. Since θ_1 is proportional to θ_2 we have from Eq. (5)

$$\delta\rho = a \sin(\theta_2 - \omega_2) + b\theta_2 + k'$$

where a, b and k' are constants, the last two including any slow uniform variation such as might result from stresses in the apparatus. Letting computed values $\delta\rho_i$ correspond to angles θ_i, we have according to the principle of least squares the condition that the most probable values of a and ω_2 are those for which

$$\Sigma(\delta\rho - \delta\rho_i)^2 \equiv \sum[a \sin(\theta_i - \omega_2) + b\theta_i - \delta\rho_i]^2$$

is a minimum. When account is taken of the fact that the data are distributed uniformly over the day, we infer from this condition that

$$a = (2/m)\sum_1^m \delta\rho_i \sin(\theta_i - \omega_2) + 2b\cos\omega_2$$

$$\tan\omega_2 = -\ \Sigma\delta\rho_i \cos\theta_i/(\Sigma\delta\rho_i \sin\theta_i + mb).$$

The constant b can be computed by comparing mean values of $\delta\rho$ on successive days; m is a number of exposures per day, usually 48.

Incidentally, the last two equations show the importance of the procedure of interleaving the exposures so that adjacent ones on any plate are made twelve hours apart. For it is known that the photographic emulsion is subject to shrinkage which varies from plate to plate, the plates may be slightly curved, and there are probably variable stresses in the apparatus due to different weights of plates; there is also a slight effect on measured diameters due to varying densities of photograph such as would result from different treatment and sensitiveness of plates. All of these errors are evidently automatically eliminated in the process of computing, however, since each is multiplied into a sine or cosine term in θ_i and then added to a similar product into a term of opposite sign. Compensation is made also for the greater shrinkage of emulsion near ends of plates, because the plates were started alternately one and two slit-widths from the end.

TABLE I.

				A	B	C	D
(1)	2.0	(25)	2.2	−2.1	−0.26	1.7	1.70
	2.4		1.6	0.1	0.03	1.5	1.49
	2.0		2.2	0.7	0.27	−1.1	−1.06
(4)	0.7	(28)	1.1	0.9	0.45	−1.7	−1.67
	0.2		1.1	−1.8	−1.10	0.0	0.00
	2.1		2.1	−1.3	−0.92	1.3	1.03
	3.2		1.4	0.8	0.63	2.8	1.98
(8)	0.9	(32)	0.6	0.0	0.00	0.6	0.37
	3.4		2.2	1.8	1.56	0.6	0.30
	2.0		0.0	1.6	1.48	2.4	0.92
	1.0		0.9	1.9	1.83	−1.7	−0.44
(12)	1.4	(36)	1.0	−0.4	−0.40	1.2	0.16
	1.3		2.1				
	2.0		0.2	$\Sigma u_i \sin \theta_i = 3.57$		$\Sigma u_i \cos \theta_i = 4.78$	
	1.7		2.1				
(16)	1.0	(40)	0.4	$\tan \omega_2 = -4.78/3.57$, $\omega_2 = 207°$			
	0.8		1.1				
	0.3		1.3	$\sin \omega_2 = -80$, $\cos \omega_2 = +0.60$			
	1.2		2.5				
(20)	−0.2	(44)	0.7	$a = (2/48)\ (0.60 \times 3.57 + 0.80 \times 4.78) = 0.26$			
	1.6		0.3				
	1.5		0.6				
	1.7		2.4				
(24)	0.8	(48)	2.7				

A sample of data for a period of three days and the computations for the resultant amplitude and phase of the sine curve to which it most closely conforms is given in Table I. The numbers in the two columns at the left are means of the three values of $\delta\rho$ at the same hour of each day, arranged in chronological order. Column A contains sums and differences of the four terms in the previous columns for which the sines of the corresponding phase angles are equal or opposite.[4] Column B contains products of the terms of

[4] The summation in the formula above for the amplitude can evidently be expanded as follows:

$$\sum_1^m \delta\rho_i \sin(\theta_i - \omega_2) = \cos \omega_2 \sum_1^{48} \delta\rho_i \sin \theta_i - \sin \omega_2 \sum_1^{48} \delta\rho_i \cos \theta_i$$

$$= \cos \omega_2 \sum_1^{12} (\delta\rho_i + \delta\rho_{25-i} - \delta\rho_{24+i} - \delta\rho_{49-i}) \sin \theta_i$$

column A into the sines of the corresponding phase angles. Columns C and D contain the corresponding quantities to A and B, using the cosine instead of the sine.

The results for the daily effect are summarized in Table II. The column headed ω contains the phase angles corresponding to the sidereal time of the maximum value of δn. The amplitudes are expressed in thousandths of a fringe.

TABLE II.

Time of year	Weighted amplitude	ω
January	0.16	89°
April	0.27	273
May	0.18	18
July	0.14	43
August	0.30	128
October	0.22	183

Since the total velocity of the earth could vary during the year by no more than twice the orbital velocity it is probably as well to average these results without reference to the first term in Eq. (5) that is, by simply adding them vectorially. When that is done, the amplitude of the resulting sine curve is 0.06 ± 0.05. Substituting in (5) this is found to correspond to a velocity $V_\beta = 24 \pm 19$ kilometers per second.[5]

SEARCH FOR LONG PERIOD EFFECT

Because the apparatus in its final form appeared to be permanently in adjustment and the average values of the ring diameters proved to be nearly constant, it became feasible to test whether an effect exists due to orbital motion, i.e., to determine the coefficient V_α in first term bracketed in Eq. (5). The direct way of doing this would evidently be like that for daily effects which has just been discussed, i.e., to determine δn for a large part of a year and fit the data to a curve of the required form. Instead, a modification of this procedure was adopted in order to make it unnecessary to keep all the experimental conditions the same for long times. It is based on the assump-

$$-\sin\omega_2\sum_1^{12}(\delta\rho_i-\delta\rho_{25-i}-\delta\rho_{24+i}+\delta\rho_{49-i})\cos\theta_i.$$

The terms in parentheses in the last two summations are the quantities in the columns A an C, respectively.

[5] There is a superficial appearance that this result conflicts with the assumption introduced in Eq. (5), i.e., that v_1 and v_2 are negligible in comparison with v_α and v_β, since the value just determined for v_β is even less than the orbital velocity v_1. The contradiction is merely apparent however; it arises from the adoption of a corresponding velocity as a means of expressing the accuracy of the result, as has been customary in discussions of the Michelson-Morley experiment. From that experiment it is not inferred that the velocity of the earth is but a few kilometers per second, but rather that the dimensions of the apparatus vary very nearly as required by relativity. From the present experiment we similarly infer that the frequency of light varies conformably to the theory.

tion that the most probable rate of variation of δn (computed from measured values of $\delta \rho$ over short times) is equal to the derivative of the most probable first term in Eq. (5). Each of three series of data, taken for periods varying from eight days to a month, and at intervals of three months, was used to compute the daily rate of change of $\rho\delta$ at those times of year. This rate was found by averaging arithmetically the readings of each day of a given series and determining by the method of least squares the slope of the most probable straight line represented by them. Similarly the most probable sine curve corresponding to these three derivatives is computed. Some 300 exposures comprised the three series.

The three computed rates of change were 0.050 ± 0.020, 0.007 ± 0.013 and -0.015 ± 0.021, all expressed in thousandths of a fringe per day. The computed sine curve has an amplitude of 2.96 thousandths and this corresponds to a velocity $V_\alpha = 15 \pm 4$ km per sec. Since the relatively small probable error is based only on the internal consistency of the data and is therefore not to be taken very seriously, this result can scarcely be regarded as indicating a real velocity. Furthermore the direction of the computed velocity is 123° away from that computed above.

As we have used only 300 exposures in the application of this method, it is evident that the accuracy could be increased by a large factor if data were taken steadily for a few months. The proverbial brevity of life, however, argues against laboring the point.

If the last result and that for the rotational effect are given the same weight and combined vectorially (ignoring difference of direction of V_α and V_β[6] their resultant is 10 ± 10 km per sec. In view of relative velocities amounting to thousands of kilometers per second known to exist among the nebulae, this can scarcely be regarded as other than a clear null result; it is of the same order of precision as that of the Michelson-Morley experiment. It is perhaps best expressed as at present in terms of a velocity, although of course the conclusion to be drawn is that the frequency of a spectral line varies in the way required by relativity.[7] This appears to be the only investigation in which a quantum phenomenon is shown to conform to Einstein's theory.

Insofar as the radiating atom may be regarded as a typical clock, the result of this experiment can be combined with assumption (b) to derive the Lorentz-Einstein transformations. Throughout the foregoing discussion we have dealt with time regarded as measured only at a fixed place in the moving system S; in order to specify unambiguously the time at another point of S it is necessary to specify the operations which define it. Perhaps the most natural meaning to attach to the concept is that the time at any point is the indication of a clock which has been moved with infinitesimal velocity to the

[6] The two results can be combined only by making some approximation.

[7] It is of course altogether possible that there is a real (inherently observable) velocity which is so nearly perpendicular to the orbital and equatorial planes as to have components in them small enough to have escaped observation, but the probability seems small in view of the nebular velocities mentioned above.

point, and from the same location as an identical clock with which it was originally in agreement; it turns out that this definition is equivalent to that of synchronizing by means of light signals.

We have shown that the frequency ν' of an atom moving with velocity v bears the relation $\nu' = (1-v^1/c^2)^{1/2}\nu$ to that of a fixed atom. Let us assume that the indications of clocks under similar conditions bear the same ratio. Suppose that at time $t=t'=0$, the origins of parallel coordinates in S and S' (previously defined) coincide, and that the S-clock passes through the origin with a small velocity with respect to S. Because of this motion, the velocity of the clock with respect to S' will have components, say, $v+u_x$, u_y, u_z; hence the times t' and t indicated by a clock in S' and the clock in S will thereafter stand in the relation

$$t = t'\left[1 - \frac{(v + u_x)^2 + u_y{}^2 + u_z{}^2}{c^2}\right]^{1/2} = \left[t'^2\left(-\frac{v^2}{c^2}\right) - \frac{2t'^2u_xv}{c^2} - t'^2u^2\right]^{1/2}.$$

Now u_xt' is equal to S'-measure of distance x traversed in S by the clock; hence $u_xt' = (1-v^2/c^2)^{1/2}x$. If this is substituted in the second term of the right side of the above equation and u is made to approach zero,

$$t = \left[t'^2\left(1 - \frac{v^2}{c^2}\right) - \frac{2vxt'}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2}\right]^{1/2}$$

and so

$$t' = \frac{1}{(1 - v^2/c^2)^{1/2}}\left[\frac{vx}{c^2} + t\left(1 + \frac{v^2}{c^4}\frac{x^2}{t^2}\right)^{1/2}\right].$$

Here x/t is the velocity of clock with respect to S, and it approaches zero with u; hence the coefficient of t in the last expression is unity, and

$$t' = [1/(1 - v^2/c^2)^{1/2}][t + (v/c^2)x]. \tag{11}$$

The statement that the systems are in uniform relative velocity, together with fact that t' is independent of y and z implies

$$x' = x'(x + vt)\ ;\ \text{hence } \partial x'/\partial x = (1/v)(\partial x'/\partial t). \tag{12}$$

The measurement in S' of the lenth of an interval δs in S is obtained by observing the distance $\delta s'$ between points in S' with which the ends of the interval coincide at same S'-time. For measurement along x' axes we have, because of Lorentz-Fitzgerald contraction

$$\delta x' \equiv \frac{\partial x'}{\partial x}\delta x + \frac{\partial x'}{\partial t}\delta t = \left(1 - \frac{v^2}{c^2}\right)^{1/2}\delta x$$

when

$$\delta t' \equiv \frac{1}{(1 - v^2/c^2)^{1/2}}\left(\delta t + \frac{v}{c^2}\delta x\right) = 0,\ \text{i.e.,}$$

when $\delta t = -(v/c^2)\delta x$. Hence

$$\frac{\partial x'}{\partial x}\delta x - \frac{v}{c^2}\,\frac{\partial x}{\partial t}\delta x = \left(1 - \frac{v^2}{c^2}\right)^{1/2}\delta x.$$

From this and (12)

$$\frac{\partial x'}{\partial x} = \frac{1}{(1 - v^2/c^2)^{1/2}} \quad \text{and} \quad \frac{\partial x'}{\partial t} = \frac{v}{(1 - v^2/c^2)^{1/2}}\,.$$

Hence

$$x' = (1 - v^2/c^2)^{-1/2}(x + vt).$$

This equation and (11) together with $y' = y$, $z' = z$, are the Lorentz-Einstein transformations; because they are known to possess the group property, the system S' which has been used as a tentative standard of reference evidently loses all trace of uniqueness.

The research set forth in this paper has been carried on over a period of several years, during which many obligations have been incurred. Preliminary work on it served as basis for the senior author's doctoral thesis at Johns Hopkins University. The main work was done at the California Institute of Technology with the aid of fellowships granted by the National Research Council, the Guggenheim Memorial Foundation and the Institute; it was completed during leave of absence granted by the University of Washington. Particularly acknowledgment is made to Professors E. T. Bell, R. C. Tolman and R. A. Millikan, whose interest and encouragement have made the work possible, and to Mr. Julius Pearson to whom several essential refinements of the apparatus are due.