

# Measurement of the Newtonian Constant of Gravitation by Precision Displacement Sensors

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Newtonian constant of gravitation  $G$  has relative uncertainty larger than other fundamental constants with some discrepancies in values between different measurements. We report a new scheme to measure  $G$  by detecting the position of test masses in precision displacement sensors induced by a force modulation from periodically rotating weights. To seek different kinds of experimental setups, laser interferometers for the gravitational wave detection and optically levitated microspheres are analyzed. High sensitivity of the gravitational wave detectors to displacement is advantageous to have a high precision in the  $G$  measurement, resulting in a relative precision of  $10^{-6}$  with a few hours of measurement time, whereas the tunability of parameters in optically levitated microspheres can enable the competitive measurement with a smaller scale setup dedicated to the  $G$  measurement. These measurements can provide an alternative method to measure  $G$  precisely, potentially leading to the improvement in the accuracy of  $G$ , as well as a better constraint on the non-Newtonian gravity at a length scale of  $\sim 1$  m.

Measurement of fundamental constants of physics has been one of the most important work in the field of metrology. Over time, more and more precise measurements have been achieved [1], and precise measurements affected the definition of the units [2]. Typically, these constants are measured with relative precisions around  $10^{-7}$  or better. The Newtonian constant of gravitation  $G$  has the worst relative precision of  $4.7 \times 10^{-5}$  [1], leaving room for improvement. One of the reasons for the large uncertainty is discrepancies by a few standard deviations between different measurements. The classical device to measure  $G$  is the torsion balance [3–13], as Cavendish used in 1798 [14]. Other methods such as measuring weight change [15], atom interferometry [16–18], and displacement measurements with an optical cavity [19–21] have also been performed. All of these measurements so far have had uncertainties of at least 10 ppm [1, 13], and discrepancies in the value remain not only between different methods but also between the same methods performed by different collaborations.

To improve the accuracy of the measurement, removing the discrepancies between different measurements is essential. The discrepancies are presumably caused by systematic errors that are not well estimated, which can have different sources for different experiments. One method to remove the discrepancies is to have a completely different setup that is potentially immune to the systematic errors previous measurements might have overlooked. In this paper, we propose a new way of measuring  $G$  using precision displacement sensors. The method is to measure periodic motion of a test mass that is driven by oscillating weights, and specifically, we analyze gravitational wave detectors and optically levitated

microspheres. The system is conceptually simple to analyze systematic errors, and has a potential for  $10^{-6}$  level precision. The setup is also compatible with the test of non-Newtonian gravity.

Figure 1 shows the setup. It consists of a test mass and two weights, all of which are assumed to be point masses for simplicity. The test mass of mass  $m$  is an object whose displacement is monitored by a displacement sensor that generates a harmonic trap with a trapping frequency of  $\omega_0$  and a damping constant of  $\gamma$  for the test mass. Weights of mass  $M$  are on the opposite end of a diameter of a circle on which the weights rotate at an angular velocity of  $\omega = 2\pi f$ . The distance between the test mass and the center of the circle is  $d$ .

The force  $\mathbf{F} = (F_x, F_y)$  on the test mass due to the gravitational attraction by the weights is calculated from the law of universal gravitation, with  $k$  defined as  $k = d/r$ .

$$F_x = \frac{GMm}{r^2} \left[ \frac{k - \cos \omega t}{(1 + k^2 - 2k \cos \omega t)^{3/2}} + \frac{k + \cos \omega t}{(1 + k^2 + 2k \cos \omega t)^{3/2}} \right] \quad (1)$$

$$F_y = \frac{GMm}{r^2} \sin \omega t \left[ \frac{1}{(1 + k^2 - 2k \cos \omega t)^{3/2}} + \frac{1}{(1 + k^2 + 2k \cos \omega t)^{3/2}} \right] \quad (2)$$

The time dependent position of the test mass is calculated by numerically integrating the equation of motion by the fourth order Runge-Kutta method. Initial position is set as the average position of the test mass determined approximately by the average force on the test mass by the weights to avoid a damped oscillation of the harmonic oscillator from being too large compared to the amplitude of the vibration induced by the weights, together with the initial velocity of zero. In case non-

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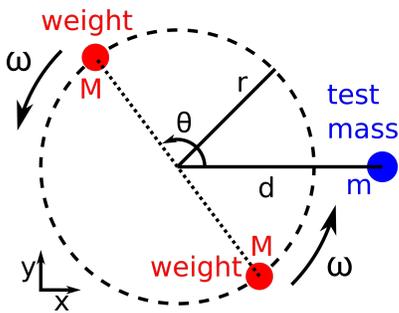


FIG. 1. Experimental setup: a pair of weights of mass  $M$  are rotating at an angular velocity of  $\omega$  on a circle of radius  $r$  on each end of a line of  $2r$  length. Test mass of mass  $m$  whose position is monitored by a sensor is located at  $d$  away from the center of the circle.

negligible amount of damped oscillation remains, initial position is adjusted manually to remove it.

The fast Fourier transformation of the position of the test mass has higher order harmonics, because the force onto the test mass is not sinusoidal. Even with a reasonable estimate of the initial position, small amount of relaxation of the test mass position to the equilibrium point is unavoidable, which appears in the frequency domain as a smooth Lorentzian function. To remove the contribution of this damped oscillation from the signal, geometrical mean of the two adjacent bins of the signal bin is subtracted from the value of the signal bin. The Fourier transformed signal is compared with the sensitivity of the displacement sensors. Although motions in both  $x$  direction and  $y$  direction can be used for the measurement, only  $x$  axis is used in the following analysis. This does not lose generality, since  $k \geq 1$  and therefore  $F_x \gtrsim F_y$ .

First, the gravitational wave detector is analyzed. As a representative of a couple of different detectors [22–24], Advanced LIGO [22] is used. To reduce unwanted perturbation to the detector system, it is assumed that the weights are located at the far end of an arm. The test mass is the cavity mirror of  $m = 40$  kg [22]. The  $x$  axis is set along the cavity axis, and the weights are on the far side of the cavity system. The motion of the mirror in  $x$  direction is governed by the horizontal pendulum mode, meaning that  $\omega_0 = 2\pi \times 0.65$  Hz and  $\gamma \sim 2\pi \times 0.065$  Hz, which is derived from the quality factor of  $Q \sim 10$  [25]. The remaining free parameter of the system is  $M$ ,  $\omega$ ,  $r$ , and  $d$ . Here,  $M = 100$  kg is assumed.

Because there are two weights, basic behavior of the test mass is an oscillation at a frequency of  $2f$ . When  $f$  increases, the signal decreases proportional to  $f^{-2}$  (see Fig. 2). This behavior is true for all the harmonics, and higher harmonics has smaller signal, as Fig. 3 shows. The comparison between Fig. 2 and the sensitivity curve of Advanced LIGO [22] tells that the signal around the

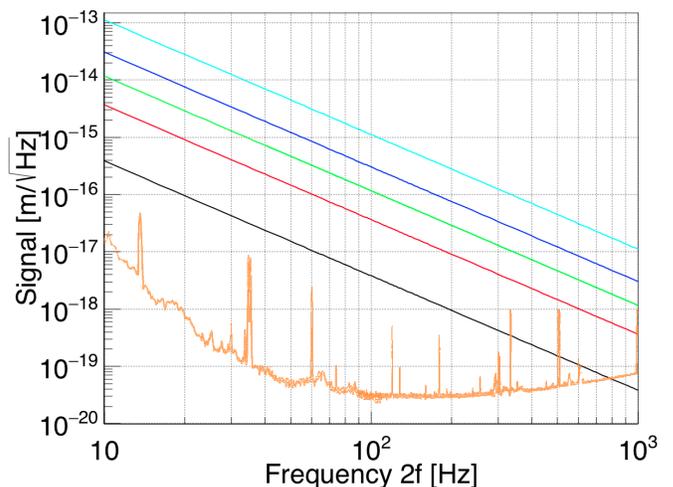


FIG. 2. Amount of signal at the frequency of  $2f$  for different rotation frequency  $f$  of the weights: different straight lines from bottom to top show the signal at  $r = 1, 3, 5, 7,$  and  $9$  m.  $d$  is fixed at  $10$  m. Orange curve at the bottom is the sensitivity curve of LIGO detector cited from Fig. 5 of Ref. [22].

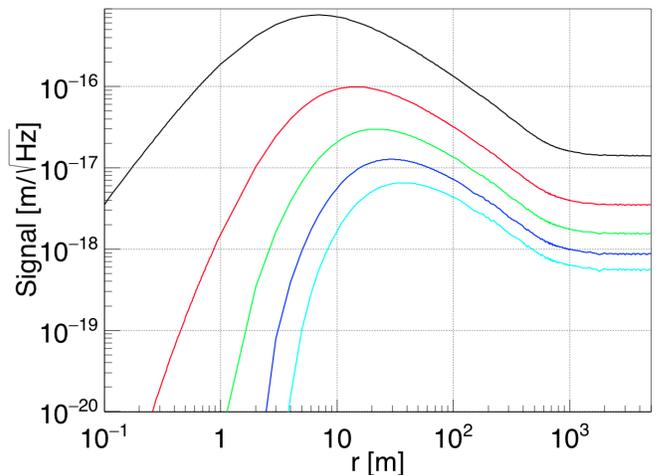


FIG. 3. Behavior of higher harmonics when  $r$  is changed: different lines from top to bottom show the signal at the frequency of  $2f, 4f, 6f, 8f,$  and  $10f$ .  $d = r$  and  $f$  are fixed at  $5$  m and  $20$  Hz, respectively.

frequency  $2f \simeq 40$  Hz has the best signal-to-noise ratio (S/N). Also, at  $f \simeq 20$  Hz, a few higher harmonics signals are also above the noise level of Advanced LIGO. When  $r = 5$  m and  $d = 10$  m, S/N is more than 1 for up to 6th harmonics.

When  $d$  is fixed, larger  $r$  gives larger signal (Fig. 2). This is because the difference between the minimum and maximum distance between a weight and the test mass ( $d - r$  and  $\sqrt{d^2 + r^2}$ , respectively) is larger when  $d - r$  is smaller. The minimum distance  $d - r$  is practically

determined by the size of the objects around the test mass, such as the test mass and the weights themselves with finite size, the damping system, and the vacuum chamber. In the following discussion, it is assumed that  $d - r = 5$  m is doable, but smaller  $d - r$  increases the signal significantly.

When  $d - r$  is fixed, remaining parameter is  $r$ . Figure 3 shows the amount of signal for different  $r$  with  $d - r = 5$  m and  $f = 20$  Hz. When  $r$  is small, the difference between the maximum force  $F_{\max}$  and the minimum force  $F_{\min}$  on the test mass is small, and therefore the signal is small. In the limit of  $d \gg r$ , the force is approximately

$$F_x = 2 \frac{GMm}{d^2} \left( 1 + \frac{9}{2} \frac{d^2}{r^2} + 6 \frac{d^2}{r^2} \cos 2\omega t \right), \quad (3)$$

which is purely sinusoidal. This explains small higher harmonic signals at small  $r$ . At the limit of  $r \rightarrow \infty$ ,  $F_{\max} - F_{\min}$  is large, but the velocity of a weight to pass the closest point to the test mass is high, and overall work on the test mass by the weights becomes small. Also, most of the force applied on the test mass occurs when the weight is closest to the test mass, i.e.  $\theta \ll 1$ . Since  $\cos \theta = \cos \omega t$ ,

$$F \simeq \frac{GMm}{d^2} \frac{1}{2\omega t} \frac{1}{(1 + (\omega t)^2/2)^{3/2}}, \quad (4)$$

when  $\omega t \ll 1$ . This is independent of  $r$ , but still the force is a function of  $\omega t$ , and therefore higher harmonic signals converge to constants. Thus, there is an intermediate  $r$  that maximizes the signal, which is around  $r = 8$  m. On one hand, larger higher harmonics helps determining  $G$  more precisely, because we can use more than one signal frequency. On the other hand, higher harmonic signals are at least factor of few smaller than that of the basic harmonics, and therefore optimizing  $r$  for obtaining large signal for higher harmonics might not necessarily benefit a lot.

The basic harmonic signal is significantly larger than the noise level of LIGO. Take  $r = 5$  m,  $d = 10$  m,  $f = 20$  Hz, and  $M = 100$  kg Hz as an example. The signal is  $7.3 \times 10^{-16}$  m/ $\sqrt{\text{Hz}}$ , which results in  $S/N = 7.3 \times 10^3$ . With this amount of signal, only  $\sim 200$  s integration time reaches the same precision as currently most precise measurements, and in a few hours, the precision increases to  $10^{-6}$ . If the measured displacement agrees with the theoretical estimate up to  $10^{-6}$  precision, the assumption of the inverse square law is correct to the same precision. This serves as a test of non-Newtonian gravity. Assuming the following modification by a Yukawa potential term with dimensionless magnitude  $\alpha$  and length scale  $\lambda$ , the expected sensitivity in  $\alpha - \lambda$  space is depicted in Fig. 4, which is beyond the current best constraint at  $\lambda \gtrsim 1$  m.

$$V(r) = \frac{GMm}{r} \left( 1 + \alpha e^{-r/\lambda} \right) \quad (5)$$

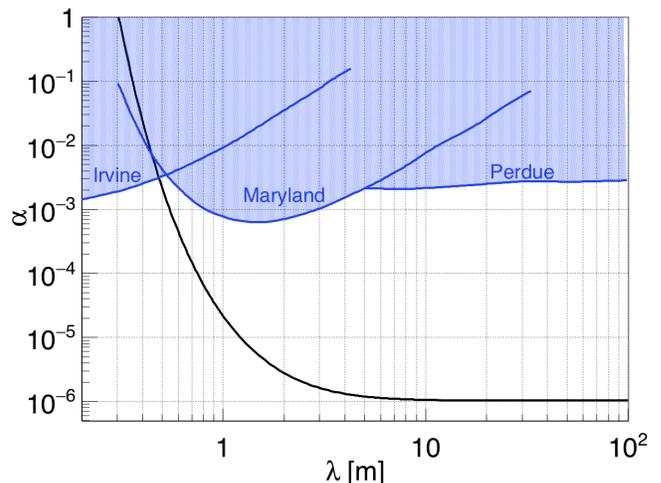


FIG. 4. Potential constraint on non-Newtonian gravity by the system described in Fig. 1 with  $10^{-6}$  precision force measurement (black curve); blue curves with shaded region are current limit adopted from Ref. [26]. Original data comes from Ref. [27] (Irvine), Ref. [28] (Maryland), and Ref. [29] (Purdue).

The argument so far suggests that  $8 \lesssim r \lesssim 20$  m is optimal. The practical limit for  $r$  is determined by the power of the motor that moves the turn table where the weights are mounted. At the parameters of  $d = 10$  m,  $r = 5$  m,  $f = 20$  Hz, and  $M = 100$  kg, total kinetic energy of the weights is 40 MJ. This is not unattainable, but very high. (cf. airplane:  $\sim 1$  GJ, car:  $\lesssim 1$  MJ) The drag force by the air is  $1.2 \times 10^4$  N, with an assumption that the weight is made of a lead sphere, resulting in the power of 7.8 MW to maintain the angular velocity. This is an order of magnitude larger than a high-power car engine, and therefore it is essential to move the weights in a vacuum or making a wind shield to reduce the drag force. Potentially, it is more realistic to reduce the  $r$  or  $f$ .  $r = 1$  m requires small kinetic energy and power by a factor of 25, with a scarification of the signal by a factor of 3.

The optically-levitated spheres are another system suitable for precision displacement sensors. Although they have sensitivities to the displacement around  $10^{-11}$  m/ $\sqrt{\text{Hz}}$  [30–33], poorer than the gravitational wave detectors, they have advantages of compact sizes and tunability of parameters for the harmonic traps,  $\omega_0$  and  $\gamma$ .  $\omega_0/2\pi$  varies from 20 Hz [34] to a few kHz [33].  $\gamma$  is typically set around the same frequency as  $\omega_0$  to keep residual noise small, but alternating cycles of strong and weak feedback cooling can achieve low noise but small  $\gamma$  situation. The dependence of the signal on  $f$  for different  $\omega_0$  is shown in Fig. 5. The signal is a standard harmonic oscillator driven at a frequency of  $2f$ .

The amount of the signal at the resonant frequency

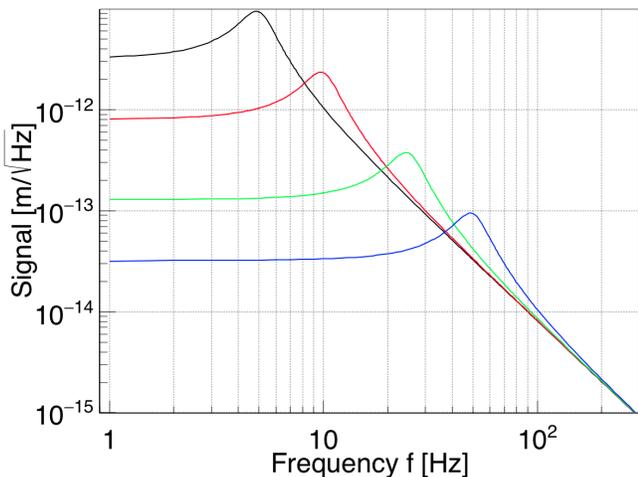


FIG. 5. Amount of signal at the frequency of  $2f$  for different rotation frequency  $f$  of the weights: different lines from bottom to top show the case of  $\omega_0/2\pi = 10, 20, 50,$  and  $100$  Hz.  $d$  and  $r$  are fixed at  $0.6$  m and  $0.3$  m, respectively.  $Q$  is fixed at  $3$ .

increases linearly to  $Q$ . The optimal value for  $Q$  is determined by a compromise between the amount of the signal, the amount of noise, and the time required for cooling and equilibration. At the mechanical resonance, not only the signal but also noise increases, compared to off-resonant frequencies. To remove noise, the measurement sequence can include a periodic cooling stage with lower  $Q$ , during which the measurement is not optimal but noise is reduced to the noise floor. At high  $Q$ , the time necessary for the driven oscillation to reach the large equilibrium amplitude can be too long to keep the noise level low. This puts an upper limit on  $Q$ . These factors should give an optimal  $Q$ , which is at least  $3$ , as the typical operation of Ref. [30] is performed at  $Q = 3$ .

Other than the resonance, the behavior of the optically-levitated microsphere is the same as the case of the gravitational wave detectors. Above the resonant frequency, lower  $f$  leads to higher signal, meaning that a system with lower resonant frequency gives higher signal. This motivates construction of a low trapping frequency system. Even at  $\omega_0 = 2\pi \times 10$  Hz, the position sensitivity of the currently available system [30] has at best S/N of  $1$ . Further reduction of noise is desired for precise measurements, and if it decreases to  $10^{-13}$  m/ $\sqrt{\text{Hz}}$ , which is the shot noise limit of radial directions in Ref. [30], S/N becomes  $100$ . With this, integration time of  $10^6$  s, which is a couple of weeks, brings the similar precision to the currently available best precision.

Other geometrical configuration can improve the measurement. Although a single weight would not be a good choice due to the large vibration at frequency  $f$  generated by the mass imbalance, four or larger even number

mass can be beneficial. On one hand, this reduces the signal, because  $F_{\max} - F_{\min}$  decreases, but on the other hand, reduced  $f$  to obtain a signal at the same frequency is advantageous to reduce the energy required to drive the rotation of the weights. Because the kinetic energy scales  $\sim f^2$ , increasing the number of weights by a factor of  $n$  reduces the total kinetic energy to  $1/n$ . An advantage specific to the optically-levitated spheres is the force measurement is performed three-dimensionally [35]. This allows the use of  $F_y$ , which potentially help increasing the precision of the measurement and discriminating background. If the trapped sphere is off the plane of the weights' rotation,  $F_z$  is also non-zero, and this also provides signal.

The discussion so far only described the ratio between the signal and noise, and background has not been considered yet. Because the signal is a continuous oscillation at frequency  $2f$ , all of transient backgrounds are removed by taking long enough measurement and picking up a specific bin in the frequency space. Monitoring environmental vibrations and vetoing the data acquisition when the environmental vibration is large also works. If necessary, it is even possible to put rotating weights at each LIGO site to take coincidence of the two independent detectors to reject phase-incoherent signal as background, in the same way as the gravitational wave detection.

The most difficult background to remove is the vibration induced by the motion of the weights. The structure is ideally symmetric relative to the center of the circular orbit, but finite mechanical tolerance can induce the shift of the center of the mass of two weights from the center of the circle. This generates mechanical oscillations at frequency  $f$ , and nonlinear response of the holding structure easily generates the vibration at  $2f$ . Measures to isolate this vibration from the detector, such as using the holding structure of the weights completely different from the detector with some damping material in between, should be taken. Also, the displacement of the weights due to the vibration itself needs to be well monitored to detect the relative change of the position by  $1 \mu\text{m}$ , which is  $\lesssim 10^{-6}$  of the distance scale between the weight and the test mass. This should be possible with a proper interferometric position sensor. Another uncertainty can come from geometrical imperfection. Precision machining typically has a precision of submicron, so to the extent that  $r$  and  $d$  are in the order of  $0.1$  m or larger, the relative uncertainty can be suppressed to  $\sim 10^{-6}$ . The precision of mass measurement can also be as high as  $10^6$ . Overall, precise measurement of  $G$  down to  $10^{-6}$  level seems possible.

To summarize, the precision detection of the displacement of the test mass is a promising new method of measuring the Newtonian constant of the gravitation  $G$ . With the state-of-the-art gravitational wave detectors such as LIGO and a pair of  $100$  kg mass rotating on a circular trajectory of radius  $5$  m, whose center is

10 m away from the test mass mirror, the relative precision of  $10^{-6}$  can be reached in a few hours of integration time. The optically levitated microspheres need certain amount of improvement in the sensitivity, and if the noise level decreases to  $10^{-13}$  m/ $\sqrt{\text{Hz}}$ , few weeks of integration time can provide the similar precision as the current best measurement.

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