Some Spherical Trigonometry

Scalar and Vector Products

In this document we will make use of the dot and vector products. If

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{1}$$

then the dot product (or scalar product) is defined by

$$\mathbf{p} \cdot \mathbf{q} = p_1 q_1 + p_2 q_2 + p_3 q_3. \tag{2}$$

We can show that $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}|\,|\mathbf{q}| \cos\omega$ where ω is the angle between $\mathbf{p}$ and $\mathbf{q}$. The dot product is a scalar and does not depend on the coordinate system used to compute it. The vector product (also known as the cross product) is

$$\mathbf{p} \times \mathbf{q} = \begin{pmatrix} p_2 q_3 - p_3 q_2 \\ p_3 q_1 - p_1 q_3 \\ p_1 q_2 - p_2 q_1 \end{pmatrix}. \tag{3}$$

This is a vector quantity (the components transform as a vector under a rotation of the axes). Geometrically the vector product $\mathbf{p} \times \mathbf{q}$ is a vector of length $|\mathbf{p}|\,|\mathbf{q}| \sin\omega$ with $\omega \in [0, \pi)$ and in a direction orthogonal to $\mathbf{p}$ and $\mathbf{q}$ given by a right-handed rule (so that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$). Note that $\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}$ and $\mathbf{p} \times \mathbf{q} \cdot \mathbf{p} = \mathbf{p} \times \mathbf{q} \cdot \mathbf{p} = 0$. The triple scalar product is

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \mathbf{p} \times \mathbf{q} \cdot \mathbf{r} = \mathbf{q} \times \mathbf{r} \cdot \mathbf{p} = \mathbf{r} \times \mathbf{p} \cdot \mathbf{q} = [\mathbf{q}, \mathbf{r}, \mathbf{p}] = [\mathbf{r}, \mathbf{p}, \mathbf{q}] \tag{4}$$

and it changes sign if we swap the order of a pair of vectors in the product: $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = -[\mathbf{q}, \mathbf{p}, \mathbf{r}]$. Finally, the triple vector product formula asserts that

$$(\mathbf{p} \times \mathbf{q}) \times \mathbf{r} = \mathbf{r} \times (\mathbf{q} \times \mathbf{p}) = (\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{r} \cdot \mathbf{q})\mathbf{p} \tag{5}$$

and thus

$$(\mathbf{p} \times \mathbf{q}) \cdot (\mathbf{r} \times \mathbf{s}) = ((\mathbf{p} \times \mathbf{q}) \times \mathbf{r}) \cdot \mathbf{s} = ((\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{r} \cdot \mathbf{q})\mathbf{p}) \cdot \mathbf{s} = (\mathbf{p} \cdot \mathbf{r})(\mathbf{q} \cdot \mathbf{s}) - (\mathbf{q} \cdot \mathbf{r})(\mathbf{p} \cdot \mathbf{s}). \tag{6}$$

Spherical Triangles

We will use spherical polar coordinates on the unit sphere, so that

$$\mathbf{x} = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} \tag{7}$$

represents a point where $\theta \in [0, \pi]$ is measured from the z-axis (co-latitude), and $\phi \in (-\pi, \pi]$ is the azimuthal angle (longitude). We set up the spherical triangle by defining the vertices on the unit sphere.

For convenience, we take the first point **a** to be the North pole and **b** along the great circle $\phi = 0$ (Greenwich meridian). Thus

$$\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ \sin c \\ \cos c \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} \sin b \cos A \\ \sin b \sin A \\ \cos b \end{pmatrix}. \tag{8}$$

It follows that **b** subtends an angle c at the centre (and hence the distance from **a** to **b** along the great circle is c). Similarly, **b** is along the great circle $\phi = A$, and its distance along this great circle from **a** is b.

We can compute the triple scalar product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (\mathbf{b} \times \mathbf{c})_3 = b_1 c_2 - b_2 c_1 = -\sin b \sin c \sin A. \tag{9}$$

But $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ does not depend on the coordinate system (it's a scalar) and, moreover, it is cyclic in **a**, **b** and **c**, so we also have that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -\sin a \sin b \sin C = -\sin c \sin a \sin B \tag{10}$$

leading to the sine rule for spherical triangles:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{11}$$

Now let us compute the lengths of the edges, we have

$$\cos a = \mathbf{b} \cdot \mathbf{c} = \cos b \cos c + \sin b \sin c \cos A \tag{12}$$

and trivially $\cos b = \mathbf{c} \cdot \mathbf{a}$ and $\cos c = \mathbf{a} \cdot \mathbf{b}$. By permuting the points cyclically we establish the relationships

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \qquad \text{and} \qquad \cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{13}$$

The Dual Triangle

Next, we construct a formula involving only one edge length. If **a**, **b** and **c** are the locations of the vertices of a spherical triangle on the unit sphere, so that $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$, then define

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|}, \qquad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{|\mathbf{c} \times \mathbf{a}|} \qquad \text{and} \qquad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}. \tag{14}$$

These are dual to the original vectors (so $\mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = 0$, i.e., **a** is a pole for which the associated equator contains the arc from **b** to **c**. Notice that the magnitudes of the vector products are

$$|\mathbf{b} \times \mathbf{c}| = \sin a, \qquad |\mathbf{c} \times \mathbf{a}| = \sin b \qquad \text{and} \qquad |\mathbf{a} \times \mathbf{b}| = \sin c. \tag{15}$$

If a' is the distance between $\mathbf{b}'$ and $\mathbf{c}'$ on the unit sphere's surface then

$$\cos a' = \mathbf{b}' \cdot \mathbf{c}' = \frac{(\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{a} \times \mathbf{b})}{\sin b \sin c} = \frac{(\mathbf{c} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{c})}{\sin b \sin c} = -\frac{\cos a - \cos b \cos c}{\sin b \sin c} = -\cos A \tag{16}$$

by Eq. (12). Thus $a' = \pi - A$, and similarly $b' = \pi - B$ and $c' = \pi - C$.

Next compute the angles of the triangle. The angle A' at the vertex $\mathbf{a}'$ is found by evaluating the dot product of the projections of $\mathbf{b}'$ and $\mathbf{c}'$ into the plane orthogonal to $\mathbf{a}'$. Call $\boldsymbol{\beta}$ projection of $\mathbf{b}'$ orthogonal to $\mathbf{a}'$, so that $\boldsymbol{\beta} = \mathbf{b}' - (\mathbf{b}' \cdot \mathbf{a}')\mathbf{a}' = \mathbf{a}' \times (\mathbf{b}' \times \mathbf{a}')$, then

$$\frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} = \frac{\mathbf{a}' \times (\mathbf{b}' \times \mathbf{a}')}{|\mathbf{a}' \times (\mathbf{b}' \times \mathbf{a}')|} = \mathbf{a}' \times \frac{\mathbf{b}' \times \mathbf{a}'}{|\mathbf{b}' \times \mathbf{a}'|}. \tag{17}$$

Now we assume that $[\mathbf{a}, \mathbf{b}.\mathbf{c}] < 0$ by Eq. (10) (swap $\mathbf{b}$ and $\mathbf{c}$ if not). Now $(\mathbf{b}' \times \mathbf{a}')/|\mathbf{b}' \times \mathbf{a}'|$ is a unit vector orthogonal to $\mathbf{a}'$ and $\mathbf{b}'$ (so will be either $\mathbf{c}$ or $-\mathbf{c}$, but we need to know which),

$$\mathbf{b}' \times \mathbf{a}' = \frac{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{b} \times \mathbf{c})}{\sin a \sin b} = \frac{(\mathbf{c} \times \mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{c} \times \mathbf{a} \cdot \mathbf{b})\mathbf{c}}{\sin a \sin b} = -\frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{c}}{\sin a \sin b} \tag{18}$$

which is a positive multiple of $\mathbf{c}$ so that

$$\frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} = \mathbf{a}' \times \mathbf{c}. \tag{19}$$

Similarly, $\boldsymbol{\gamma}$ the projection of $\mathbf{c}'$ onto the plane orthogonal to $\mathbf{a}'$, is $\boldsymbol{\gamma} = \mathbf{c}' - (\mathbf{c}' \cdot \mathbf{a}')\mathbf{a}'$. Compute

$$\frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} = \mathbf{a}' \times \mathbf{c} = \frac{(\mathbf{b} \times \mathbf{c}) \times \mathbf{c}}{\sin a} = \frac{(\mathbf{b} \cdot \mathbf{c})\mathbf{c} - \mathbf{b}}{\sin a} = \frac{\mathbf{c} \cos a - \mathbf{b}}{\sin a} \tag{20}$$

Likewise,

$$\frac{\boldsymbol{\gamma}}{|\boldsymbol{\gamma}|} = \mathbf{a}' \times (-\mathbf{b}) = \frac{\mathbf{b} \cos a - \mathbf{c}}{\sin a} \tag{21}$$

Thus

$$\cos A' = \frac{\boldsymbol{\beta} \cdot \boldsymbol{\gamma}}{|\boldsymbol{\beta}|\,|\boldsymbol{\gamma}|} = \frac{(\mathbf{b} \cos a - \mathbf{c}) \cdot (\mathbf{c} \cos a - \mathbf{b})}{\sin^2 a} = \frac{(1 + \cos^2 a)(\mathbf{b} \cdot \mathbf{c}) - 2 \cos a}{\sin^2 a} = -\cos a, \tag{22}$$

since $\mathbf{b} \cdot \mathbf{c} = \cos a$. In consequence, $A' = \pi - a$. Similarly $B' = \pi - b$ and $C' = \pi - c$. It follows from Eq. (12) applied to the dual triangle that

$$\cos A = -\cos a' = -[\cos b' \cos c' + \sin b' \sin c' \cos A'] = \sin B \sin C \cos a - \cos B \cos C. \tag{23}$$

By cyclic permutation we find:

$$\cos B = \sin C \sin A \cos b - \cos C \cos A \qquad \text{and} \qquad \cos C = \sin A \sin B \cos c - \cos A \cos B. \tag{24}$$

For a sphere of radius R we replace $a \mapsto a/R$, $b \mapsto b/R$ and $c \mapsto c/R$ in all these formulae, so in particular we have the rule:

$$\cos\left(\frac{c}{R}\right) = \frac{\cos C + \cos A \cos B}{\sin A \sin B}. \tag{25}$$